THE GENERALIZED THIN-SANDWICH PROBLEM AND ITS LOCAL SOLVABILITY

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Abstract

We consider Einstein Gravity coupled to matter consisting of a gauge field with any compact gauge group and coupled scalar fields. We investigate under what conditions a free specification of a spatial field configuration and its time derivative determines a solution to the field equations (thin-sandwich problem). We establish sufficient conditions under which the thin-sandwich problem can be solved locally in field space.

Introduction

In this paper we consider the initial value problem for Einstein gravity plus matter in spacetimes $\Sigma \times \mathbf{R}$, where Σ is a closed orientable 3-manifold. We are interested in the question of how to find initial data which satisfy the constraints. The most popular approach here is a powerful method devised by Lichnerowicz, Choquet-Bruhat, York and others, henceforth referred to as the "conformal method". (See [I] for a brief review and [CY] for more details.) Of the gravitational variables it allows to freely specify the conformal class of the initial 3-metric, the conformally rescaled transverse-traceless components of the extrinsic curvature and a constant trace thereof (i.e. Σ must have constant mean curvature). Given these data, the constraints turn into a quasilinear elliptic system of second order for the conformal factor (scalar function) and the transverse momentum (vector field), which decouples due to the constant mean-curvature condition. The disadvantages of this

1

method are that it does not easily generalize to data of variable mean curvature and that it does not allow to control the local scales of the physical quantities initially, since the freely specifiable data (gravitational and non-gravitational) are related to the actual physical quantities by some rescalings with suitable powers of the conformal factor. In particular, one has no control over the conformal part of the initial 3-geometry.

In this paper we are concerned with the so called "thin-sandwich method", which differs from the one just mentioned insofar as it aims to define solutions to the Einstein equations by a *free* specification of the initial field configuration and its coordinate-time-derivative. The constraints are now read as equations for the gauge parameters (lapse, shift, ..). From the conformal point of view this means that one tries to trade in the freedom to specify the gauge parameters for the freedom to specify the conformal part of the metric and the longitudinal part of the momentum. The disadvantages mentioned above would then be overcome, but unfortunately the equations (for the gauge parameters) turn out to be non-elliptic[1] in general [BO][CF]. However, for certain open subsets of initial data they are elliptic and can be locally solved. This was first shown in [BF] and will be shown in a wider context with dynamical matter here.

We note that historically this approach arose from the question (then formulated as a conjecture) of whether the specification of two 3-geometries uniquely determine an interpolating Einstein space-time (thick-sandwich problem). For nearby geometries infinitesimally close in time this turns into the thin-sandwich problem (see [W] (chapter 4) and [BSW]) which we now describe in more detail.

In a space-time neighborhood $\Sigma \times \mathbf{R}$ of the Cauchy surface Σ, we use the standard parametrization of the space-time metric $g^{(4)}$,

$$g^{(4)} = -\alpha^2 \, dt \otimes dt + g_{ab}(dx^a + \beta^a dt) \otimes (dx^b + \beta^b dt), \qquad (1.1)$$

where g is the (t-dependent) Riemannian metric of Σ and α, β are (t-dependent) scalar- and vector fields on Σ, known as lapse and shift. The extrinsic curvature reads

$$K = \tfrac{1}{2\alpha}(\partial_t - L_\beta)g, \qquad (1.2)$$

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[1] By ellipticity of non-linear differential operators one means the ellipticity of its linearization, which depends on the point (in field space) about which one linearizes. The usual statement that the thin-sandwich equations are not elliptic merely asserts the existence of points where the linearization is not elliptic, but not that the domain of ellipticity is empty.

where L_β denotes the Lie-derivative along β. Written in terms of (g, K), the constraints do not depend on α and β and hence constrain the set of allowed values for the data (g, K). The constraints read

$$K_{ab}K^{ab} - (K_a^a)^2 - R = -2T_{\perp\perp}, \tag{1.3}$$

$$\nabla^b(K_b^a - \delta_b^a K_c^c) = T_\perp^a, \tag{1.4}$$

where T is the energy-momentum tensor of the matter and $\perp$ denotes the component along the future pointing normal n of Σ.

Alternatively, one may write the constraints in terms of g and $\dot{g} := \partial_t g$ by replacing K via (1.2). Then they explicitly involve α and β and one may ask whether it is possible to *freely* specify $(g, \dot{g})$ and let the constraints determine α and β. This is the thin-sandwich problem (TSP). If we abbreviate $\Psi := (g, \dot{g})$, $X := (\alpha, \beta)$, the constraints take the form of the thin-sandwich equation (TSE):

$$F[\Psi, X] = 0. \tag{1.5}$$

The TSP now asks for existence and uniqueness of solutions of the TSE, read as equation for X given Ψ. Once this is solved, we can construct (g, K) satisfying (1.3-4), which uniquely determine space-time via the Einstein evolution equations [CY].

It has long been discussed in the literature that, in general, existence and uniqueness should fail [BO], although some arguments merely showed this under the additional assumption (a priori) of constant lapse function α [CF] (see also [G] for a related issue). However, more recently it was shown that given a solution (Ψ, X) of the TSE which satisfies certain bounds on geometric quantities and which admits no nontrivial solutions of the spatially projected Killing equation, there exist unique solutions $X'(\Psi')$ for all Ψ' in a neighbourhood of Ψ. This was achieved by an implicit-function-theorem for a reduced version of (1.5) with already eliminated lapse function.

Note that in this formulation of the TSP the right hand sides of the constraints, that is $T_{\perp\perp}$ and $T_\perp^a$, are assumed given. These are *not* the components of T that an observer along ∂_t would measure. The relation between the two sets of components involve α and β. For example, if the matter is represented by some dynamical field ϕ, the quantities $T_{\perp\perp}$ and $T_\perp^a$ cannot be calculated from the initial data $(\phi, \dot{\phi})$ without the use of α and β. Hence there is a certain inconsistency in the traditional

formulation of the TSP, in that it eliminates any appearance of the normal n in favour of ∂_t, lapse and shift on the left side (the gravitational part), but not on the right side (matter part) of the constraints. This inconsistency was already felt by others (see e.g. section IV of [BO]), but no alternative formulation was hitherto attempted.

In this paper we consider a generalized thin-sandwich problem (GTSP) for the initial data of the full system of coupled gravitational and matter fields, which avoids the difficulty just mentioned. As matter we shall consider coupled systems of scalar and gauge fields, further specified below. By Φ we shall collectively denote all dynamical fields of the theory. We ask: Under what conditions do input data $\Psi := (\Phi, \dot{\Phi})$ uniquely specify a solution to the Einstein-matter equations? In the same fashion as for (1.5), one obtains a now *generalized* thin-sandwich equation (GTSE) from which one tries to determine the "gauge parameters" X given the data Ψ. One non-trivial aspect of our generalization is due to the possible presence of gauge matter fields. In this case X comprises lapse, shift and *additional* functions with values in the Lie-algebra of the gauge group. Our main result will consist of an implicit-function-theorem for this extended set of variables, which for our GTSP is precisely analogous to the result proven in [BF] for the traditional form of the thin-sandwich problem. But note that the two formulations differ even without gauge fields.

The Generalized Framework

The dynamical fields we consider involve the gravitational field, a gauge field with compact gauge group **G** of dimension N and an M-component scalar field with values in an associated $\mathbf{R}^M$-vector-bundle. It couples to both previous fields in the standard minimal fashion. For simplicity we assume the **G**-principal bundle to be trivial. Since the frame bundle of any orientable 3-manifold is always trivial, we may once and for all choose global trivialisations of these bundles and represent fields by their globally defined component-fields on space-time. For fixed time, a configuration of fields is given by the $6 + 3N + M$ component-fields on Σ,

$$\Phi^A := (g_{ab}, A_a^\mu, \phi^\alpha), \tag{2.1}$$

where indices $\mu, \nu..$ denote components in the Lie-algebra **LG** and $\alpha, \beta..$ denote

4

components in $\mathbf{R}^M$. Hence we think of a field configuration as mapping

$$\Phi : \Sigma \longrightarrow GL(3, \mathbf{R})/SO(3) \times \mathbf{R}^{3N} \times \mathbf{R}^M, \tag{2.2}$$

where we identify the space of symmetric, positive definite matrices, in which g_{ab} is valued, with the first factor space on the right hand side. The total target space, whose dimension is $6 + 3N + M$, will be denoted by Θ, and the space of mappings $\Sigma \to \Theta$ (to be further specified) by $\mathcal{M}$.

Compactness of the gauge-group $\mathbf{G}$ implies the existence of $\mathbf{G}$-invariant, symmetric, positive definite, bilinear forms $k_{\mu\nu}$ and $h_{\alpha\beta}$ on $\mathbf{LG}$ and $\mathbf{R}^M$ respectively. The class of models we shall consider here are characterized by the Lagrange four-form

$$\mathcal{L} = \tfrac{1}{2} * R^{(4)} - \tfrac{1}{4} k_{\mu\nu} \Omega^\mu \wedge *\Omega^\nu - \tfrac{1}{2} h^{\alpha\beta} \nabla\phi_\alpha \wedge *\nabla\phi_\beta - W, \tag{2.3}$$

where $*$ is the Hodge-duality map wrt. $g^{(4)}$ and Ω is the curvature of A. For notational simplicity we shall denote all the covariant derivatives acting on sections in the various vector bundles by the same symbol ∇. In general it will therefore involve the Christoffel symbols of g as well as the gauge connection A in the appropriate representation of $\mathbf{LG}$. The potential W depends on the fields and its first spatial derivatives. Its precise form is not important, except that we need to explicitly exclude 2$\underline{\mathrm{nd}}$ (or higher) derivative couplings, in particular, the so called conformal coupling of the scalar and gravitational field. The reason for this will be explained below. The Hamiltonian constraint for (2.3) has the general form

$$\mathcal{H} = \tfrac{1}{2} G_{AB} V^A V^B + U = 0, \tag{2.4}$$

where the potential is the following sum:

$$U = -R + \tfrac{1}{4} k_{\mu\nu} g^{ac} g^{bd} \Omega^\mu_{ab} \Omega^\nu_{cd} + h_{\alpha\beta} g^{ab} \nabla_a \phi^\alpha \nabla_b \phi^\beta + W, \tag{2.5}$$

whose terms represent the contributions of the gravitational, gauge, and scalar fields (two terms) respectively to the potential energy. R denotes the Ricci-scalar for g. The "canonical velocities", V^A, can be written in terms of the "coordinate velocities" $\dot{\Phi}^A := \partial_t \Phi^A$ and the "gauge-parameters" α and $\xi := (\beta, \lambda)$. The general structure is (compare (1.2))

$$V = \tfrac{1}{\alpha} \Gamma = \tfrac{1}{\alpha} \left(\dot{\Phi} + f_\xi \right), \tag{2.6}$$

where f_ξ represents the motion generated by the infinitesimal diffeomorphism and gauge-transformation with parameters β and λ respectively. Resolved in terms of the individual fields (2.1), the components of Γ, and hence of f_ξ, read:

$$\Gamma_{ab} = \dot{g}_{ab} - 2\nabla_{(a}\beta_{b)}\,, \tag{2.7}$$

$$\Gamma_a^\mu = \dot{A}_a^\mu - \beta^b\Omega_{ba}^\mu - \nabla_a\lambda^\mu\,, \tag{2.8}$$

$$\Gamma^\alpha = \dot{\phi}^\alpha - \beta^a\nabla_a\phi^\alpha + \lambda^\mu\rho_{\mu\beta}^\alpha\phi^\beta\,, \tag{2.9}$$

where ρ denotes the representation of **LG** in $gl(M, \mathbf{R})$.

Finally, following the decomposition of Θ as cartesian product, the "kinetic-energy-metric" G_{AB} on Θ which appears in (2.4) has the following block-structure:

$$G_{AB} = G^{ab\,cd} \oplus k_{\mu\nu}g^{ab} \oplus h_{\alpha\beta}, \tag{2.10}$$

where the first 6×6 block is given by the DeWitt-metric

$$G^{ab\,cd} = \tfrac{1}{4}\left(g^{ac}g^{bd} + g^{ad}g^{bc} - 2g^{ab}g^{cd}\right), \tag{2.11}$$

which is a Lorentz metric of signature (1,5). Hence G_{AB} itself is a Lorentz metric of signature $(1, 5+3N+M)$ on the manifold Θ, which is homeomorphic to $\mathbf{R}^{6+3N+M}$. We shall sometimes denote this metric simply by G and write $G(\cdot,\cdot)$ for the inner product. We will see that the Lorentzian signature of G is *the* important feature on which the proofs of our main results rest. This is also the reason why we had to exclude higher derivative (e.g. conformal) couplings of the scalar and gravitational fields, since they will in general destroy this signature structure [K]. On the other hand, our proofs will still apply to more complicated self-couplings of the scalar field. For example, non-linear σ-models would be allowed, since here the target space metric of the scalar field, $h_{\alpha\beta}$, simply becomes ϕ^α-dependent, which is unimportant to our proofs as long as it stays positive definite.

The (undensitized) momenta of the field Φ^A are just given by the covariant components – with respect to G – of the velocities: $P_A := G_{AB}V^B$. For the individual fields we write $P_A = (\pi^{ab}, \pi_\mu^a, \pi_\alpha)$. In the canonical theory, the phase space function that generates infinitesimal diffeomorphisms and gauge-transformations with parameter ξ' is given by

$$\mathcal{P}_{\xi'} := \int_\Sigma d\mu\, P_A f_{\xi'}^A, \tag{2.12}$$

6

where here and below we set $d\mu := \sqrt{\det\{g_{ab}\}}d^3x$ for the measure on Σ. For completeness we remark that, using (2.7-9), a straightforward calculation yields the following familiar Poisson-bracket relation, which involves the curvature tensor Ω of the gauge field on Σ:

$$\{\mathcal{P}_{\xi'}, \mathcal{P}_{\xi''}\} = P_{\xi'''}, \tag{2.13}$$

where $\xi''' = (\beta''', \lambda''')$ reads

$$\beta''' = -[\beta', \beta''], \tag{2.14}$$

$$\lambda''' = [\lambda', \lambda''] - \Omega(\beta', \beta''). \tag{2.15}$$

The diffeomorphism- and Gauß-constraints are just given by $\mathcal{P}_{\xi'} = 0 \,\forall\xi'$, which may be expressed by saying that the velocity field V is L^2G-orthogonal to all "vertical" vector fields $f_{\xi'}$. Hence we must have

$$0 = \int_\Sigma P_A f_{\xi'}^A \, d\mu = \int_\Sigma \tfrac{1}{\alpha}G(\Gamma, f_{\xi'}) \, d\mu =: \int_\Sigma (g_{ab}\beta'^a \mathcal{D}^b + k_{\mu\nu}\lambda'^\mu \mathcal{G}^\nu) \, d\mu, \tag{2.16}$$

for all, say C^∞, vector fields β' and **LG**-valued functions λ'. This is equivalent to $\mathcal{D}^a = 0$ (diffeomorphism constraint) and $\mathcal{G}^\mu = 0$ (Gauß constraint). Explicitly we get

$$\mathcal{D}^a = 2G^{abcd}\nabla_b\left(\tfrac{1}{\alpha}\Gamma_{cd}\right) - \tfrac{1}{\alpha}g^{ab}k_{\mu\nu}\Omega^\mu_{bc}g^{cd}\Gamma^\nu_d - \tfrac{1}{\alpha}g^{ab}(\nabla_b\phi^\alpha)h_{\alpha\beta}\Gamma^\beta, \tag{2.17}$$

$$\mathcal{G}^\mu = \nabla_a\left(\tfrac{1}{\alpha}g^{ab}\Gamma^\mu_b\right) + \tfrac{1}{\alpha}k^{\mu\nu}\rho^\alpha_{\nu\beta}\phi^\beta h_{\alpha\gamma}\Gamma^\gamma. \tag{2.18}$$

Given Φ and $\dot\Phi$ we now have the $4 + N$ equations $0 = \mathcal{H} = \mathcal{D}^a = \mathcal{G}^\mu$ for the $4 + N$ unknowns $\alpha, \beta^a, \lambda^\mu$. The first step consists of inserting (2.6) into (2.4) and solving for α^2:

$$\alpha^2 = -\frac{G(\Gamma, \Gamma)}{2U}. \tag{2.19}$$

For this to make sense the right-hand side must be positive. But for the following analysis it turns out that we need to put the following stronger

Condition 1 (a priori):

$$U > 0, \tag{2.20}$$

$$G(\Gamma, \Gamma) < 0. \tag{2.21}$$

7

Note that (2.20) just involves the initial data, whereas (2.21) contains as well the unknowns β and λ (hence "a priori"). Note that (2.21) says that the system must, at each point of Σ, move in a "timelike" direction with respect to the Lorentz metric G. The need for such an a priori bound implies that our results will only be perturbative. Given these bounds, we set α equal to the positive square root of the right hand side of (2.19). We can then eliminate α from (2.17),(2.18) and obtain a set of $3 + N$ equations for the $3 + N$ unknowns $\xi = (\beta, \lambda)$, which we call the generalized *reduced* thin-sandwich equation (GRTSE):

$$F[\Psi, \xi] = 0. \tag{2.22}$$

Now consider the following functional over configurations satisfying Condition 1:

$$S[\Psi, \xi] = \int_{\Sigma} \sqrt{-2U\, G(\Gamma, \Gamma)}\, d\mu, \tag{2.23}$$

then solutions to the GRTSE are stationary points with respect to variations in ξ. More precisely, let D_2 denote the partial (functional-) derivative in the second argument and denote the L^2 inner product of ξ_1 and ξ_2 by $\langle \xi_1 | \xi_2 \rangle := \int_{\Sigma} d\mu(k_{\mu\nu}\lambda_1{}^{\mu}\lambda_2{}^{\nu} + g_{ab}\beta_1^a\beta_2^b)$. We then have

Lemma 1.

$$D_2 S[\Psi, \xi](\xi') = -\langle \xi' | F[\Psi, \xi] \rangle \tag{2.24}$$

PROOF. For $s \in (-\epsilon, \epsilon)$ set $\eta(s) = \xi + s\xi'$ and $\Gamma(s) = \dot{\Phi} + f_{\eta(s)}$. Hence $\frac{d}{ds}|_{s=0}\Gamma(s) = f_{\xi'}$. Then, recalling (2.16),

$$\frac{d}{ds}\Big|_{s=0} S[\Psi, \eta(s)] = -\int_{\Sigma} \sqrt{\frac{-2U}{G(\Gamma, \Gamma)}}\, G(\Gamma, f_{\xi'})\, d\mu = -\langle \xi' | F[\Psi, \xi] \rangle \tag{2.25}$$

$\square$

Main Results

In this section we are mainly concerned with the linearization of the GRTSE, except at the end where we will discuss global uniqueness. The corresponding linear operator will be called L, without explicit indication that it depends on Ψ and ξ. It is defined by

$$\langle \xi' | L\xi'' \rangle := \frac{d}{ds}\Big|_{s=0} \langle \xi' | F[\Psi, \xi + s\xi''] \rangle = \int_\Sigma d\mu \, \frac{d}{ds}\Big|_{s=0} \frac{G(f_{\xi'}, \Gamma(s))}{\alpha(s)}, \qquad (3.1)$$

where $\Gamma(s) = \dot{\Phi} + f_{\xi + s\xi''}$ and $\alpha(s) = [-G(\Gamma(s), \Gamma(s))/2U]^{1/2}$. Setting $\Gamma(s = 0) = \Gamma$, $\alpha(s = 0) = \alpha$, and noting that $\frac{d}{dt}\big|_{s=0}\Gamma(s) = f_{\xi''}$, we get

$$\frac{d}{ds}\Big|_{s=0}\alpha(s) = -\frac{G(f_{\xi''}, \Gamma)}{2\alpha U} = \alpha \frac{G(f_{\xi''}, \Gamma)}{G(\Gamma, \Gamma)}, \qquad (3.2)$$

where we used (2.4) to eliminate U in the last step. Hence

$$\frac{d}{ds}\Big|_{s=0} \frac{G(f_{\xi'}, \Gamma(s))}{\alpha(s)} = \frac{1}{\alpha}\left[G(f_{\xi'}, f_{\xi''}) - \frac{G(f_{\xi'}, \Gamma)G(f_{\xi''}, \Gamma)}{G(\Gamma, \Gamma)}\right] = \frac{G(f_{\xi'}^\perp, f_{\xi''}^\perp)}{\alpha}, \quad (3.3)$$

where

$$f_{\xi'}^\perp := f_{\xi'} - \Gamma\frac{G(f_{\xi'}, \Gamma)}{G(\Gamma, \Gamma)} \qquad (3.4)$$

is the G-orthogonal projection of $f_{\xi'}$ perpendicular to Γ. This leads to the following expression for L's matrix elements:

$$\langle \xi' | L\xi'' \rangle = \int_\Sigma d\mu \, \tfrac{1}{\alpha} G(f_{\xi'}^\perp, f_{\xi''}^\perp), \qquad (3.5)$$

where α is the square-root of the rhs. of (2.19). If (2.21) holds, Γ is "timelike" (pointwise on Σ) and hence the $f^\perp$'s are "spacelike" or zero. Since the metric G is Lorentzian, it is positive definite on "spacelike" vectors. Hence we have shown

Lemma 2. *Suppose Condition 1 holds, then L is self-adjoint and non-negative. Furthermore, $\xi' \in kernel(L) \Leftrightarrow \exists \kappa : \Sigma \to \mathbf{R}$ such that*

$$f_{\xi'} = \kappa\Gamma. \qquad (3.6)$$

Remark: Symmetry is expected, for consider the rhs. of (2.23) as functional of Ψ and Γ, denoting it by $S[\Psi, \Gamma]$, then the calculation of the rhs. of (3.5) was just that of $-D_2^2 S[\Psi, \Gamma](\Gamma', \Gamma'')$ for $\Gamma' = f_{\xi'}$ and $\Gamma'' = f_{\xi''}$.

We will next show that under the same hypotheses L is in fact elliptic. For this we will need

Lemma 3. *If Condition 1 holds, π^{ab} is a positive or negative definite matrix.*

PROOF. Condition 1 implies $G^{AB}P_A P_B < 0 \Rightarrow G_{ab\,cd}\pi^{ab}\pi^{cd} = 2(\pi_{ab}\pi^{ab} - \frac{1}{2}(\pi_a^a)^2) < 0$. Choosing a frame where $g_{ab} = \delta_{ab}$ and $\pi^{ab} = \mathrm{diag}\,(p_1, p_2, p_3)$, this is equivalent to the following condition on the eigenvalue-vector $\vec{p} := (p_1, p_2, p_3)$:

$$\left|\frac{\vec{n}\cdot\vec{p}}{\|\vec{p}\|}\right| > \sqrt{\frac{2}{3}}, \tag{3.7}$$

where $\vec{n} = (1,1,1)/\sqrt{3}$, which means that $\vec{p}$ lies in the interior of the double cone with axis along $\vec{n}$ and opening angle $\theta < \cos^{-1}(2/3)^{1/2}$ about its axis. This cone just touches the walls of the positive and negative octants along the bisecting lines, and since $\vec{p}$ must be in its interior, all eigenvalues are either strictly positive or strictly negative. $\qquad\square$

Proposition 1. *Suppose Condition 1 holds, then the second order differential operator L is elliptic.*

PROOF. We shall calculate the principal symbol of L. For this we need to go back to the explicit formulae (2.17) and (2.18) for the full non-linear problem and explicitly linearize them, but keeping track only of the highest (second) derivatives. By $\overset{2}{=}$ we shall denote equality in the second derivative terms. We set again $\Gamma(s) = \dot{\Phi} + f_{\xi + s\xi'}$ etc., with $\xi' = (\beta', \lambda')$. It will be convenient to express things in terms of the momenta, using $\alpha P_A = G_{AB}\Gamma^B$, and accordingly write (3.2) in the form

$$\frac{d}{ds}\Big|_{s=0}\alpha(s) = -\frac{P_A f_{\xi'}^A}{2U}. \tag{3.8}$$

Then

$$\frac{d}{ds}\Big|_{s=0}\mathcal{D}^a \overset{2}{=} \partial_b \left[\frac{P_A f_{\xi'}^A}{\alpha U}\pi^{ab} - \frac{4}{\alpha}G^{ab\,cd}\partial_c \beta'_d\right]$$

$$\overset{2}{=} -\frac{1}{\alpha}\left[\frac{2\pi^{ac}\pi^{de}g_{db}\partial_c\partial_e\beta'^b + \pi^{ab}\pi_\nu^c\partial_b\partial_c\lambda'^\nu}{U} + g^{bc}\partial_b\partial_c\beta'^a - g^{ac}\partial_c\partial_b\beta'^b\right] \tag{3.9}$$

$$\frac{d}{ds}\Big|_{s=0}\mathcal{G}^\mu \overset{2}{=} \partial_a \left[\frac{P_A f_{\xi'}^A}{2\alpha U}k^{\mu\nu}\pi_\nu^a - \frac{1}{\alpha}g^{ab}\partial_b\lambda'^\mu\right]$$

$$\overset{2}{=} -\frac{1}{\alpha}\left[\frac{2k^{\mu\nu}\pi_\nu^a g_{bc}\pi^{cd}\partial_a\partial_d\beta'^b + k^{\mu\sigma}\pi_\sigma^a\pi_\nu^b\partial_a\partial_b\lambda'^\nu}{2U} + g^{ab}\partial_a\partial_b\lambda'^\mu\right]. \tag{3.10}$$

Replacing $\partial_a \to k_a$ we can just read-off the matrix of the principal symbol $\sigma(k)$ in the general form

$$\left[\begin{array}{c|c} \sigma^\mu_\nu & \sigma^\mu_b \\ \hline \sigma^a_\nu & \sigma^a_b \end{array}\right],$$

(3.11)

where we have chosen to order the $N + 3$ rows and columns so that we first count the N components of λ' and then the 3 components of β'. In order to calculate the determinant we make the following simplifications: We call $\pi^{ab}k_b =: p^a$, $\pi^a_\mu k_a =: \pi_\mu$, $\pi^\mu := k^{\mu\nu}\pi_\nu$ and choose a spatial frame where $g_{ab} = \delta_{ab}$, $k_a = (\|k\|, 0, 0)$ and $p^a = (p_1, p_2, 0)$. Then (3.11) reads explicitly

$$\sigma(k) = -\frac{1}{\alpha}\left[\begin{array}{c|ccc} \frac{\pi^\mu \pi_\nu}{2U} + \|k\|^2 \delta^\mu_\nu & \frac{\pi^\mu p_1}{U} & \frac{\pi^\mu p_2}{U} & 0 \\ \hline \frac{p_1 \pi_\nu}{U} & \frac{2p_1^2}{U} & \frac{2p_1 p_2}{U} & 0 \\ \frac{p_2 \pi_\nu}{U} & \frac{2p_1 p_2}{U} & \frac{2p_2^2}{U} + \|k\|^2 & 0 \\ 0 & 0 & 0 & \|k\|^2 \end{array}\right].$$

(3.12)

Now, for $k \neq 0$, $p_1 = \|k\|\pi(\hat{k}, \hat{k})$, where $\hat{k} := k/\|k\|$. Lemma 3 then ensures that $p_1 \neq 0$. In order to calculate $\det\{\sigma(k)\}$, we simplify this matrix as follows: We subtract $\frac{\pi^\mu}{2p_1}$ times the $N + 1^{\underline{\text{st}}}$ row from the $\mu^{\underline{\text{th}}}$ row, for each $1 \leq \mu \leq N$, and also subtract $\frac{p_2}{p_1}$ times the $N + 1^{\underline{\text{st}}}$ row from the $N + 2^{\underline{\text{nd}}}$ row. The resulting matrix reads

$$-\frac{1}{\alpha}\left[\begin{array}{c|ccc} \|k\|^2 \delta^\mu_\nu & 0 & 0 & 0 \\ \hline \frac{p_1 \pi_\nu}{U} & \frac{2p_1^2}{U} & \frac{2p_1 p_2}{U} & 0 \\ 0 & 0 & \|k\|^2 & 0 \\ 0 & 0 & 0 & \|k\|^2 \end{array}\right].$$

(3.13)

Its determinant, which equals that of $\sigma(k)$, is now easily calculated:

$$\det\{\sigma(k)\} = \frac{2p_1^2}{U}\|k\|^{2(N+2)}\left[-\frac{1}{\alpha}\right]^{N+3} = 2\left[-\frac{\|k\|^2}{\alpha}\right]^{N+3}\frac{[\pi(\hat{k}, \hat{k})]^2}{U}.$$

(3.14)

Lemma 3 implies that this is zero $\Leftrightarrow k = 0$, which finally proves ellipticity of L. $\square$

The results obtained so far suffice to deduce an implicit-function-theorem. To state it precisely, we need to choose appropriate function spaces. It is natural to choose Sobolev spaces since they are also used to show existence for the time evolution [CY]. To begin with, it is convenient to summarize the order of spatial

11

differentiation by which the various fields enter the quantities Γ, U, α and hence the GRTSE, by the following matrix:

	Γ	U	α	GRTSE
g_{ab}	1	2	2	3
A_a^μ	1	1	1	2
ϕ^α	1	1	1	2
$\dot{g}_{ab}$	0	–	0	1
$\dot{A}_a^\mu$	0	–	0	1
$\dot{\phi}^\alpha$	0	–	0	1
β^a	1	–	1	2
λ^μ	1	–	1	2

$$(3.15)$$

Note that we assumed that U only contained first derivatives of the matter-fields, whereas it contains second derivatives of the gravitational field through the Ricci-scalar. Hence g_{ab} enters the GRTSE thrice differentiated.[2]

By $H^n(V)$ we shall denote the Sobolev space of V-valued functions on Σ with L^2-norm in the first n derivatives (i.e. generalizing $H^n = W^{n,2}$ using an inner product in V). We shall have $V = T_2^0$ for g_{ab} and $\dot{g}_{ab}$, $V = T_1^0 \otimes \mathsf{LG}$ for A_a^μ and $\dot{A}_a^\mu$, $V = \mathbf{R}^M$ for ϕ^α and $\dot{\phi}^\alpha$, $V = T_0^1$ for β^a and $V = \mathsf{LG}$ for λ^μ. The inner products for the various V's are just as in the metric G, except for the gravitational field where instead of $G^{ab\,cd}$, which is not positive definite, we choose the positive definite form $g^{ac}g^{bd}$ (compare (2.11)). Now we define the Sobolev spaces

$$H_\Phi^n := H^{n+3}(T_2^0) \times H^{n+2}(T_1^0 \otimes \mathsf{LG}) \times H^{n+2}(\mathbf{R}^M), \qquad (3.16)$$

$$H_{\dot{\Phi}}^n := H^{n+1}(T_2^0) \times H^{n+1}(T_1^0 \otimes \mathsf{LG}) \times H^{n+1}(\mathbf{R}^M), \qquad (3.17)$$

$$H_\Psi^n := H_\Phi^n \times H_{\dot{\Phi}}^n, \qquad (3.18)$$

$$H_\xi^n := H^n(T_0^1) \times H^n(\mathsf{LG}). \qquad (3.19)$$

One may now show that the operator F in the GRTSE, $F[\Psi, \xi] = 0$, defines a C^1-map

$$F : H_\Psi^n \times H_\xi^{n+2} \to H_\xi^n \quad \text{for } n \geq 2 \qquad (3.20)$$

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[2] This seems to have been overlooked in [BF].

on the domain of fields (Ψ, ξ) which satisfy Condition 1.[3] For this we need to impose suitable but very mild regularity conditions on the unspecified function W in (2.5). The linear map L is the first derivative of F in wrt. the second argument: $D_2 F[\Psi, \xi]$. Ellipticity implies that $\text{Image}(L) := L(H_\xi^{n+2}) \subseteq H_\xi^n$ is closed and hence H_ξ^n splits as orthogonal sum of closed subspaces, given by L's image and the kernel of L's adjoint. Hence, since L is self-adjoint, $H_\xi^n = \text{image}(L) \oplus \text{kernel}(L)$. We now get an implicit function theorem for the map F if $D_2 F[\Psi, \xi]$ is a linear isomorphism, i.e. if $\text{kernel}(L) = \{0\}$. But since L is elliptic, any non-trivial element in the kernel may be represented by a C^∞ function ξ' which must then satisfy (3.6). Hence a trivial kernel is equivalent to the following condition for smooth functions:

Condition 2.

$$f_{\xi'} = \kappa \Gamma \text{ implies } \xi = 0, \ \kappa = 0. \tag{3.21}$$

Hence we arrive at the following formulation of an implicit-function-theorem for the generalized thin-sandwich problem. It may be seen as the analog or generalization of Theorem 2 (and 3) in [BF].

Theorem. *Let $n \geq 2$ and $(\Psi, \xi) \in H_\Psi^n \times H_\xi^{n+2}$ be a solution to the GRTSE, $F[\Psi, \xi] = 0$, which satisfies Condition 1 (i.e. (2.20-21)) and Condition 2 (i.e. (3.21)). Then there exist open neighbourhoods $V \subset H_\Psi^n$ of Ψ and $W \subset H_\Psi^n \times H_\xi^{n+2}$ of (Ψ, ξ) and a C^1-map $\sigma : V \to H_\xi^{n+2}$ such that $F[\Psi', \xi'] = 0$ for $(\Psi', \xi') \in W \Leftrightarrow \Psi' \in V$ and $\xi' = \sigma(\Psi')$.*

Consider the action T of $\dot{\mathbf{R}} := \mathbf{R} - \{0\}$ on $H_\Psi^n \times H_\xi^{n+2}$, given by $T_\delta(\Phi, \dot{\Phi}, \xi) := (\Phi, \delta\dot{\Phi}, \delta\xi)$ for $\delta \in \dot{\mathbf{R}}$. It leaves individually invariant the three subsets of points (Ψ, ξ) which 1.) obey Condition 1, 2.) obey Condition 2, 3.) solve the GRTSE. To see this, recall that f_ξ is linear in ξ, hence $T_\delta\Gamma = \delta\Gamma$. Invariance of the first set is now obvious. Further, if $f_{\xi'} = \kappa\Gamma$ has only the trivial solution, then so does $f_{\xi'} = \kappa T_\delta\Gamma$, since otherwise $(\delta^{-1}\xi', \kappa)$ would be a non-trivial solution to the first equation. Hence the second set is invariant. Finally, since Γ scales with δ and the square-root of expression (2.19) for α with $|\delta|$, the GRTSE (2.16) changes at most by an overall sign, which proves invariance of the third set.

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[3] The Sobolev embedding theorem for 3-dimensional domains and L^2-norms implies a continuous embedding $H^n(V) \hookrightarrow C^k(V)$ for $k < n - 3/2$. $n \geq 2$ is needed to guarantee continuity of the functions and gain pointwise control, which is needed in the proof for F being C^1.

We can now repeat the Theorem for each point $T_\delta(\Psi, \xi)$ on the $\dot{\mathbf{R}}$-orbit of (Ψ, ξ) with open sets V_δ, W_δ and solution maps σ_δ. In this way the solution map σ extends to a solution map $\sigma^* : V^* \to H_\xi^{n+2}$, where $V^* := \bigcup_{\delta \in \dot{\mathbf{R}}} V_\delta$, which uniquely represents all solutions in $W^* := \bigcup_{\delta \in \dot{\mathbf{R}}} W_\delta$. By construction it satisfies $\sigma^*(\Phi, \delta\dot{\Phi}) = \delta\sigma(\Phi, \dot{\Phi}) \ \forall \delta \in \dot{\mathbf{R}}$. Dropping the superscript $*$, we formulate this as

Corollary 1. *Let (Ψ, ξ) be as in the Theorem. Then there exist open neighbourhoods $V \subset H_\Psi^n$ of $\bigcup_{\delta \in \dot{\mathbf{R}}} T_\delta \Psi$ and $W \subset H_\Psi^n \times H_\xi^{n+2}$ of $\bigcup_{\delta \in \dot{\mathbf{R}}} T_\delta(\Psi, \xi)$ and a C^1-map $\sigma : V \to H_\xi^{n+2}$ such that $F[\Psi', \xi'] = 0$ for $(\Psi', \xi') \in W \Leftrightarrow \Psi' \in V$ and $\xi' = \sigma(\Psi')$. Moreover,*

$$\sigma(\Phi, \delta\dot{\Phi}) = \delta\sigma(\Phi, \dot{\Phi}), \quad \forall \delta \in \dot{\mathbf{R}} \tag{3.22}$$

Finally we prove that Condition 2 not only ensures local but also global uniqueness. The analogous result has been proven for the traditional RTSE in [BO].[4]

Proposition 2. *Let (Ψ, ξ) and $(\Psi, \tilde{\xi})$ satisfy Condition 1 and (Ψ, ξ) the GRTSE. Then $(\Psi, \tilde{\xi})$ satisfies the GRTSE $\Leftrightarrow$ there exists a positive function $r : \Sigma \to \mathbf{R}_+$ such that $\Gamma = \dot{\Phi} + f_\xi$ and $\tilde{\Gamma} = \dot{\Phi} + f_{\tilde{\xi}}$ are related by*

$$\tilde{\Gamma} = r\Gamma. \tag{3.23}$$

PROOF. $\Leftarrow$: This follows trivially from the fact that the GRTSE, i.e. equations (2.17-18), contain ξ only through the combination $\frac{1}{\alpha}\Gamma$.

$\Rightarrow$: For $s \in [0, 1]$, consider the convex combinations $\xi(s) := s\xi + (1 - s)\tilde{\xi}$ and $\Gamma_s := \dot{\Phi} + f_{\xi(s)} = s\Gamma + (1 - s)\tilde{\Gamma}$. In the following it is useful to think of each Γ_s as section in the pulled-back bundle $\Phi^* T(\Theta)$ whose fibre at $p \in \Sigma$ is a Minkowski space $\mathbf{R}^{1,5+3N+M}$ with metric $G_{\Phi(p)}$. Condition 1 requires $\Gamma(p)$ and $\tilde{\Gamma}(p)$ to be "timelike", so that $s \mapsto \Gamma_s(p)$ is the straight path connecting these two "timelike" vectors. First we show that $\Gamma(p)$ and $\tilde{\Gamma}(p)$ lie in the interior of the same "light-cone" for some, and hence all, $p \in \Sigma$. To see this, we consider, for each p, the inner product $G(\Gamma - \tilde{\Gamma}, \mathcal{V})$ with the timelike vector $\mathcal{V} := g_{ab} \frac{\partial}{\partial g_{ab}}$ of constant length-squared $G(\mathcal{V}, \mathcal{V}) = -3$. Now,

$$\int_\Sigma G(\Gamma - \tilde{\Gamma}, \mathcal{V}) \, d\mu = 2 \int_\Sigma \nabla_a(\beta^a - \tilde{\beta}^a) \, d\mu = 0, \tag{3.24}$$

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[4] The full statement and proof given in [BO] contains an additional part which is erroneous, as was first pointed out in [BF]. If transcribed to our setting, the incorrect part would amount to the claim that (3.23) implied $r \equiv 1$.

so that, because Σ is connected, there exists a point $p \in \Sigma$ where $G(\Gamma - \tilde{\Gamma}, \mathcal{V})(p) = 0$. Hence $\Gamma(p)$ and $\tilde{\Gamma}(p)$ point in the same half of the "light-cone", and so does $\Gamma_s(p)$, since the interior of the half "light-cone" is a convex set. By continuity this must then be true at each point $p \in \Sigma$ so that $G(\Gamma_s, \Gamma_s)$ is a negative-valued function on Σ for each s.

Next we consider the function

$$I(s) := S[\Psi, \xi(s)] = \int_{\Sigma} \sqrt{-2U\, G(\Gamma_s, \Gamma_s)}\, d\mu. \tag{3.25}$$

We have $I'(0) = 0 = I'(1)$, where $' = \frac{d}{ds}$, since ξ and $\tilde{\xi}$ solve the GRTSE. Furthermore, a straightforward calculation yields:

$$I''(s) = \int_{\Sigma} \frac{[2U]^{\frac{1}{2}}}{[-G(\Gamma_s, \Gamma_s)]^{\frac{3}{2}}} G(\Gamma, \Gamma) G(\tilde{\Gamma}_{\perp}, \tilde{\Gamma}_{\perp})\, d\mu \leq 0, \tag{3.26}$$

with

$$\tilde{\Gamma}_{\perp} := \tilde{\Gamma} - \Gamma \frac{G(\Gamma, \tilde{\Gamma})}{G(\Gamma, \Gamma)}. \tag{3.27}$$

The inequality in (3.26) results from Γ being "timelike" and $\tilde{\Gamma}_{\perp}$ being "spacelike" or zero. But $I'(0) = I'(1) = 0$ and $I''(s) \leq 0$ imply $I'' \equiv 0$. On the other hand, equality in (3.26) can only be achieved for $\tilde{\Gamma}_{\perp} = 0$ which is equivalent to (3.23), where r must be positive-valued since Γ and $\tilde{\Gamma}$ point in the same half of the "light-cone". $\qquad\square$

Now (3.23) implies (3.6) with $\xi' = \tilde{\xi} - \xi$ and $\kappa = \frac{r-1}{r}$, so that Condition 2 will enforce $r = 1$ and $\xi = \tilde{\xi}$. Hence we have

Corollary 2. If (Ψ, ξ) and $(\Psi, \tilde{\xi})$ satisfy the GRTSE and Conditions 1 and 2, then $\xi = \tilde{\xi}$.

Discussion

It is obvious that the strategy of the (generalized) thin-sandwich approach cannot work for all data. Obvious bad data are those for which the Hamiltonian constraint cannot be solved for a nowhere vanishing α. For example, consider fields Φ such that $U > 0$ and velocities $\dot{\Phi}$ whose gravitational part is pure gauge: $\dot{g}_{ab} = 2\nabla_{(a}\xi'_{b)}$. The Hamiltonian constraint implies $(\nabla_a\eta^a)^2 \geq U\alpha^2$, where $\eta = \xi' - \xi$, showing that α must vanish somewhere since $\int_\Sigma d\mu \, \nabla_a\eta^a = 0$ and Σ is connected. To avoid such situations, Condition 1 or its reversed version, $U < 0$ and $G(\Gamma, \Gamma) > 0$ may be imposed. However, if the second condition is chosen formula (3.14) together with the proof of Lemma 3 show that L manifestly fails to be elliptic, thus leaving only Condition 1.

The technical Condition 2 has an interpretation in terms of the "canonical data" (Φ, V), where $\alpha V = \dot{\Phi} + f_\xi$ and where we assume $(\Phi, \dot{\Phi}, \xi)$ to satisfy Condition 1 in order to have $\alpha \neq 0$. Namely, if $f_{\xi'} = \kappa\Gamma$ for some non-zero (ξ', κ), then $f_{\xi'} = \alpha\kappa V$ says that the same canonical data admit a representation in terms of the new lapse function $\alpha_{\mathrm{new}} = \kappa\alpha$ (now possibly with zeros), gauge functions $\xi_{\mathrm{new}} = \xi'$ and coordinate-velocities $\dot{\Phi}_{\mathrm{new}} = 0$. Conversely, if an α_{new} exists such that $\alpha_{\mathrm{new}}V = f_{\xi_{\mathrm{new}}}$, then $f_{\xi'} = \kappa\Gamma$ with $\kappa = \alpha_{\mathrm{new}}/\alpha$ and $\xi' = \xi_{\mathrm{new}}$. Hence Condition 2 precisely excludes the existence of other representations of the same canonical data with vanishing coordinate-velocities $\dot{\Phi}$. We note that Condition 2 may itself be implied by simple geometric conditions on Φ. One such set of conditions is provided by the following

Proposition 3. *Condition 2 is implied by Condition 1 and the following conditions on Φ:*

$$(i) \quad Ric < 0 \ (Ric = \ Ricci\text{-}tensor\ of\ g) \tag{4.1}$$

$$(ii) \quad \nabla_a\lambda^\mu = 0 \ and \ \lambda^\mu\rho^\alpha_{\mu\beta}\phi^\alpha = 0 \ imply \ \lambda^\mu = 0 \tag{4.2}$$

PROOF. Let $f_{\xi'} = \kappa\Gamma$, then $G(\Gamma, \Gamma) < 0 \Rightarrow G(f_{\xi'}, f_{\xi'}) \leq 0 \Rightarrow$

$$\int_\Sigma d\mu \left[G^{ab\,cd}4\nabla_{(a}\beta'_{b)}\nabla_{(c}\beta'_{d)} \right] = \int_\Sigma d\mu \, 2 \left[\nabla_{[a}\beta'_{b]}\nabla^{[a}\beta'^{b]} - R_{ab}\beta'^a\beta'^b \right] \leq 0 \tag{4.3}$$

$\Rightarrow \beta' = 0$. Since G is positive definite on $f_{\xi'}$'s for which $\beta' = 0$, $G(f_{\xi'}, f_{\xi'}) \leq 0$ implies $f_{\xi'} = 0$ which for $\beta' = 0$ is equivalent to the first two equations in (4.2). $\square$

We recall that metrics with $Ric < 0$ exist on any 3-manifold Σ [GY] (e.g. quite in contrast to $Ric > 0$, which is well known to imply a finite fundamental group). (4.2) should be read as a mild genericity-condition for the matter fields. For example, if we have a single $U(1)$ gauge field and a charged scalar field (here represented by a real doublet (ϕ^1, ϕ^2)) then condition (4.2) is satisfied if the scalar field is not identically zero.

Finally we comment on the functional $S[\Psi, \xi]$ defined in (2.23). Given that Condition 1 is satisfied, solutions to the GRTSE are stationary points with respect to variations in ξ (Lemma 1). Lemma 2 asserts that these must be minima which are stable if Condition 2 is satisfied. We now assume Conditions 1 and 2 to hold and evaluate $S[\Psi, \xi]$ at a solution $\xi = \sigma(\Psi)$. We get a C^1-function $S_* : H_\Psi^n \to \mathbf{R}_+$,

$$S_*[\Phi, \dot{\Phi}] = \int_\Sigma d\mu \sqrt{-2U \, G \left(\dot{\Phi} + f_{\sigma(\Phi, \dot{\Phi})}, \dot{\Phi} + f_{\sigma(\Phi, \dot{\Phi})} \right)}, \tag{4.4}$$

which satisfies

$$S_*[\delta\Psi] = |\delta| S_*[\Psi] \tag{4.5}$$

for all $\delta \in \dot{\mathbf{R}}$. Standard consequences are

$$D_2 S_*[\Phi, \delta\dot{\Phi}](\dot{\Phi}) = \text{sign}(\delta) \, S_*[\Phi, \dot{\Phi}], \tag{4.6}$$

$$D_2^2 S_*[\Phi, \delta\dot{\Phi}](\dot{\Phi}, \dot{\Phi}) = 0, \tag{4.7}$$

where in (4.7) we assumed C^2-smoothness of σ. It is tempting to try and regard S_* as a kind of metric on at least an open subset of the tangent bundle of the space of fields. For pure gravity it generalizes a previously considered expression which is valid only for constant lapse function [G] and also gives rigorous meaning to a formal definition of a distance function given in [CF]. But presently it is unclear to us whether S_* indeed defines an interesting geometric structure.[5]

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[5] One may wonder whether it defined a Finsler metric. For this one would have to show that the bilinear form $D_2^2 S_*^2[\Phi, \dot{\Phi}]$ is (weakly) non-degenerate. But this is not even the case in finite dimensions for functions of the form (4.4) (i.e. sum of square-roots rather than square-root of sum). Take e.g. the function $f(y_1, \ldots, y_n) = \sum_i \sqrt{y_i^2}$. Then $\partial_i \partial_j f = 2 \partial_i f \partial_j f$, which is obviously just of rank one.

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